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L_p Approximation by Subsets of Convex Functions of Several Variables*

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If S is a bounded convex subset of R^m, the problem is to find a best approximation to a function in $L_p(S)$, $1 \leqslant p \leqslant \infty$, by an arbitrary subset of convex functions. An existence theorem for a best approximation is established under a certain condition on the subset. In particular, a best convex approximation exists. Also investigated are properties of norm-bounded subsets and L_p-convergent sequences of convex functions. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let L_p, $1 \leqslant p \leqslant \infty$, be the Lebesgue space of extended real functions on a bounded convex subset of R^m. The problem is to find a best approximation to a function in L_p by an arbitrary subset of convex functions. It is shown that, under a certain condition on the subset, a best approximation exists. In particular, a best approximation from the set of all convex functions exists. As a tool for analysis, properties of norm-bounded subsets and convergent sequences of convex functions are explored.

Let $S \subset R^m$ be a bounded convex body, i.e., a convex set with nonempty interior $\text{int}(S)$. Let H be the set of all extended real-valued functions on S. Let $L_p = L_p(S)$, $1 \leqslant p < \infty$, denote the Banach space of all (equivalence classes of) Lebesgue measurable functions f in H with $\int |f|^p < \infty$ and norm $\|f\|_p = (\int |f|^p)^{1/p}$. Similarly, $L_\infty = L_\infty(S)$ is the Banach space of (equivalence classes of) essentially bounded functions f with norm $\|f\|_\infty = \text{ess sup } |f|$. A function k in H is said to be convex if

$$k(\lambda s + (1-\lambda)\, t) \leqslant \lambda k(s) + (1-\lambda)\, k(t) \qquad (1.1)$$

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for all $0<\lambda<1$ and all $s, t \in S$ for which the right-hand side of (1.1) is defined; i.e., only those, s, t for which $k(s)$ and $k(t)$ are simultaneously not infinite with opposite signs are to be considered [11]. Equivalently, (1.1) may be considered only when $k(s)<\infty$ and $k(t)<\infty$. It can easily be shown that k is convex if and only if its epigraph,

$$E(k)=\{(s, u) \in S \times R : k(s) \leqslant u\},$$

is a convex subset of $S \times R$ [11]. Let $K \subset H$ denote the set of all convex functions. Clearly, K is a convex cone. Let $P \subset K$ be an arbitrary set. In what follows a notation such as $P \cap L_p$ denotes all equivalence classes in L_p to which a function in P belongs. As usual, we carry out the arguments for the representative element of the class. Let $f \in L_p$ and Δ denote the infimum of $\|f-k\|_p$ for k in $P \cap L_p$. The problem is to find a g in $P \cap L_p$, called a best approximation to f from $P \cap L_p$, so that $\|f-g\|_p=\Delta$. For $1<p<\infty$, L_p is uniformly convex and, hence, a best approximation from $P \cap L_p$ exists and is unique if $P \cap L_p$ is closed and convex [3]. We are interested in examining existence when $P \cap L_p$ is not necessarily convex.

We say that $P \subset H$ is a.e. sequentially closed if it is closed under a.e. convergence of sequences of functions. We denote by $\bar{P}$ the smallest superset of P which is a.e. sequentially closed. Note that P is a.e. sequentially closed if and only if $P=\bar{P}$. Our main results appear in Section 3. We show that if $P \subset H$ satisfies the condition, $P \cap L_p=\bar{P} \cap L_p$, then $P \cap L_p$ is closed in L_p and a best aproximation from $P \cap L_p$ exists. In particular, K satisfies this condition for all $1 \leqslant p \leqslant \infty$, and, hence, these results are applicable to $K \cap L_p$. The following property of bounded sequences is basic in the derivation of this result. If (k_n) is a norm-bounded sequence in $P \cap L_p$, then there exists a subsequence (g_i) of (k_n) and g in $P \cap L_p$ such that $g_i \rightarrow g$ pointwise on $\operatorname{int}(S)$ and a.e. on S. Such a sequence is bounded above on every compact $T \subset \operatorname{int}(S)$ and below on S uniformly in n. Furthermore, if (k_n) is a sequence in $K \cap L_p$ and $\|k_n-k\|_p \rightarrow 0$ for some k which is continuous on $\operatorname{int}(S)$, then $k_n \rightarrow k$ pointwise on $\operatorname{int}(S)$ and uniformly on all compact subsets $T \subset \operatorname{int}(S)$. Such a property has been shown to hold in [5] for monotone (n-convex) function defined on a bounded open real interval. In Section 2, we establish several preliminary results. The analysis of the distance function measuring the distance of a point in a convex set from its complement is of independent interest. This function is concave on the convex set, and it is a tool in the analysis of the problem.

We established in [8, 9] the existence and some properties of a best L_p-approximation from subsets of special functions on a compact real interval. The unifying treatment and results were applicable to various classes of functions including quasi-convex, convex, super-additive, star-shaped, monotone, and n-convex functions. However, the analysis used the theory of functions of bounded variation on a compact real interval. The absence of any such theory on R^m and the complications presented by the higher dimensionality require us to develop different methods. Again the lattice structure that was significant in the analysis of the isotone approximation problem [4] is not applicable to our problem, hence the methods of [4] cannot be used. The problem of uniform approximation by convex functions on $S \subset R^m$ is analyzed in [7, 10].

2. Preliminaries

We establish some preliminary results which are used later. We first introduce some notation. Recall that $\operatorname{int}(A)$ denotes the interior of $A \subset R^m$. We denote by $\bar{A}$ the closure of A, and by $B(s, r)$ and $\bar{B}(s, r)$, respectively, the open and closed balls in R^m with center s and radius r. Let μ denote the Lebesgue measure on R^m.

We note that if $A \subset R^m$ is convex, then $\operatorname{int}(A)$ is convex and $\mu(\bar{A} - \operatorname{int}(A))$, the measure of its boundary points, equals zero [1]. Recall that A is a convex body if it is convex with $\operatorname{int}(A)$ nonempty [12]. For such a set, $\bar{A} = \overline{\operatorname{int}(A)}$.

For $A \subset R^m$, define the distance function $d(s, A)$ for s in R^m by

$$d(s, A) = \inf\{|s - t| : t \in A\},$$

where $|s|$ is the Euclidean norm of s. It is known that d is Lipschitzian; i.e., for all s, t in R^m,

$$|d(s, A) - d(t, A)| \leqslant |s - t|. \tag{2.1}$$

It is easy to show that there exists t in $\bar{A}$ such that $d(s, A) = |s - t|$. If A is convex, then $\bar{A}$ is convex and such a t is unique, and d is a convex function of s [12]. In the next two propositions, we analyze $d(s, A)$ when A is not convex and obtain properties of convex sets.

Proposition 2.1. *Let $S \subset R^m$ be a bounded convex body. Then $d(s) = d(s, R^m \setminus S)$, $s \in R^m$, is a Lipschitz continuous function which is concave on $\bar{S}$ with $\{s \in S : d(s) > 0\} = \operatorname{int}(S)$. Furthermore, for $r > 0$ sufficiently small, if $T = \{s \in S : d(s) \geqslant r\}$, then T is a compact convex body with $T \subset \operatorname{int}(S)$.*

Proof. Clearly $d(s) > 0$ for s in S if and only if $s \in \operatorname{int}(S)$. We establish the following concave inequality for d: if $s, t \in S$, then

$$d(\lambda s + (1 - \lambda) t) \geqslant \lambda d(s) + (1 - \lambda) d(t), \qquad 0 \leqslant \lambda \leqslant 1. \tag{2.2}$$

Suppose first that $d(s) > 0$ and $d(t) > 0$. Then the sets $B(s, d(s))$ and

$B(t, d(t))$ are contained in $\text{int}(S)$. Since $\text{int}(S)$ is convex, the convex hull E of these two sets is contained in $\text{int}(S)$. If $u = \lambda s + (1-\lambda)\, t$, then, clearly, $B(u, \lambda d(s) + (1-\lambda)\, d(t)) \subset E$. Hence, (2.2) holds. Suppose now that $d(s) > 0$ and $d(t) = 0$. Then $s \in \text{int}(S)$ and $t \in \bar{S} \setminus \text{int}(S)$. Let F be the convex hull of $B(s, d(s))$ and $\{t\}$. Let $F' = F \setminus \{t\}$. By Theorem 6.1 of [6], because the relative interior of S equals $\text{int}(S)$, we have $\lambda s + (1-\lambda)\, t \in \text{int}(S)$ for $0 < \lambda \leqslant 1$. Hence, $F' \subset \text{int}(S)$. Then, as before, $d(u, \lambda d(s)) \subset F'$, which shows that $d(u) \geqslant \lambda d(s)$ and (2.2) holds. If $d(s) = d(t) = 0$, then clearly (2.2) holds. Lipschitz continuity follows from (2.1). By concavity and continuity of d, T is compact and convex. It is contained in $\text{int}(S)$, and, for small r, $\text{int}(T) = \{s : d(s) > r\}$ is not empty.

LEMMA 2.1. *Let S and d be as in Proposition 2.1. Then there exists a sequence (T_n) of compact convex bodies with $T_n \subset T_{n+1}$ such that $\bigcup T_n =$ $\text{int}(S)$. Furthermore, if $T' \subset \text{int}(S)$ is any compact set, then there exists a compact convex body T with $T' \subset T \subset \text{int}(S)$.*

Proof. The required T_n are given by $T_n = \{s \in S : d(s) \geqslant 1/n\}$. Define $r = \min\{d(s) : s \in T'\}$. Then $r > 0$ and $T' \subset \{s \in S : d(s) \geqslant r\} = T$.

LEMMA 2.2. *Let S and d be as in Proposition 2.1. Let (F_n) be a sequence of measurable subsets of S such that* $\limsup \mu(F_n) < \mu(S)$. *Define*

$$\delta_n = \sup\{d(s) : s \in S \setminus F_n\}.$$

Then $\liminf \delta_n > 0$.

Proof. Suppose that $\liminf \delta_n = 0$. Then there exists a decreasing convergent subsequence of (δ_n) with limit 0. Assume, without loss of generality, that $\delta_n \geqslant \delta_{n+1}$ and $\delta_n \to 0$. We show a contradiction. Define $G_n = \{s \in S : d(s) > \delta_n\}$. Then $G_n \subset G_{n+1}$, $G_n \subset F_n$, and $\bigcup G_n = \text{int}(S)$. Hence, $\mu(G_n) \to \mu\,(\text{int}(S)) = \mu(S)$. It follows that $\mu(F_n) \to \mu(S)$, a contradiction. The proof is complete.

In the next two propositions, we establish the existence of a best approximation from $P \cap L_p$ under general conditions and develop convergence properties of an equi-Lipschitzian sequence of L_p. Recall the definition of $\bar{P}$ from Section 1.

A subset F of H is called equi-Lipschitzian relative to a set $T \subset S$ if each f in F is finite on T, and for some $c > 0$,

$$|f(s) - f(t)| \leqslant c\,|s - t| \tag{2.3}$$

for all f in F and all s, t in T. We remark that the conditions on $P \cap L_p$ given in the next proposition are called the property of boundedly a.e.

sequential compactness in [2], where a general theory of existence of best approximations is developed. The proof of the proposition is similar to that of Theorem 2.7 (1) of [2] and is presented here for the convenience of the reader.

PROPOSITION 2.2. *Suppose that $P \subset H$ satisfies the following conditions:*

(i) $P \cap L_p = \bar{P} \cap L_p$.

(ii) *Every norm-bounded sequence (k_n) in $P \cap L_p$ contains a subsequence (g_i) such that $g_i \to g$ a.e. on S for some g in L_p.*

Then $P \cap L_p$ is closed in L_p and a best approximation to f in L_p from $P \cap L_p$ exists.

Proof. We prove the proposition for $1 \leqslant p < \infty$. The proof for $p = \infty$ is simpler.

To show the existence of a best approximation, let $k_n \in P \cap L_p$ with $\|f - k_n\| \leqslant \Delta + 1/n$, where Δ is defined in Section 1. Then, by condition (ii), (k_n) contains a subsequence (g_i) such that $g_i \to g$ a.e. where $g \in L_p$. Clearly, $g \in \bar{P}$ and, by condition (i), $g \in P \cap L_p$. By Fatou's Lemma [3], $\|f - g\|_p \leqslant \liminf \|f - g_i\|_p = \Delta$. Hence, g is a best approximation. If $P \cap L_p$ is not a closed set, then a function that is not in the set but is in its closure does not have a best approximation. This is a contradiction. The proof is complete.

PROPOSITION 2.3. *Let $T \subset S$ be a compact convex body. Let (k_n) be a sequence in L_p, $1 \leqslant p < \infty$, such that $\|k_n - k\|_p \to 0$ for some k in L_p. If (k_n) is equi-Lipschitzian relative to T and k is continuous on T, then $k_n \to k$ uniformly on T and k is Lipschitzian on T.*

Proof. Suppose that (k_n) satisfies (2.3) on T. We first show that $k_n \to k$ on T. Suppose $s \in T$, $\varepsilon > 0$, and $\theta = \varepsilon/(2c)$. By continuity of k at s, there exists $0 < r < \theta$ so that if $V = T \cap B(s, r)$, then $|k(s) - k(t)| \leqslant \varepsilon/2$ for all t in V. We show that $\mu(V) > 0$. Since $T = \overline{\operatorname{int}(T)}$, there exists $v \in \operatorname{int}(T) \cap B(s, r)$. Consequently, for some $\rho > 0$, $B(v, \rho) \subset V$ and, hence, $\mu(V) > 0$. Now $|k_n(s) - k_n(t)| \leqslant \varepsilon/2$ for t in V. Hence, $|k_n(t) - k(t)| \geqslant |k_n(s) - k(s)| - \varepsilon$ for all t in V, for all n. If χ denotes the characteristic function of V, then

$$\|k_n - k\|_p \geqslant \|(k_n - k)\,\chi\|_p \geqslant \max\{|k_n(s) - k(s)| - \varepsilon, 0\}\,\mu(V)^{1/p}.$$

Letting $n \to \infty$, we have $k_n(s) \to k(s)$ on T. Since k_n satisfies (2.3) on T, so does k and, thus, k is Lipschitzian. To show uniform convergence on T, we use a known argument ([3, p. 266] or [6, p. 90]). Let $\varepsilon > 0$ and $W \subset T$ be a finite set so that every element of T is at a distance no greater than $\varepsilon/(3c)$. Since T is bounded, this is possible. Again, since W is finite, there exists

$N>0$ so that $|k_n(t)-k(t)|\leqslant \varepsilon/3$ for all t in W, all $n\geqslant N$. Given s in T, let t in W satisfy $|s-t|\leqslant \varepsilon/(3c)$. Then, for all $n\geqslant N$, we have

$$|k_n(s)-k(s)|\leqslant |k_n(s)-k_n(t)|+|k_n(t)-k(t)|+|k(t)-k(s)|\leqslant \varepsilon$$

uniformly for all s in T. The proof is complete.

LEMMA 2.3. *If $k\in K\cap L_p$, $1\leqslant p\leqslant \infty$, then $k>-\infty$ on S.*

Proof. Suppose to the contrary that $k(t)=-\infty$ for some t in S. We first select, in the following manner, a set of points s_i in S, $0\leqslant i\leqslant n$, with $s_0=t$ so that s_i are affinely independent (i.e., s_i-s_0, $1\leqslant i\leqslant n$, are linearly independent) and $k(s_i)<\infty$, $i\geqslant 1$. Since $\mu(S)>0$, there exists some $s_1\in S\setminus\{s_0\}$ for which $k(s_1)<\infty$, for otherwise k is not in L_p. In general, having chosen affinely independent s_i, $0\leqslant i\leqslant j$, with $j<n$ and $k(s_i)<\infty$, let aff_j denote the affine variety or flat spanned by $\{s_i\}$. Since the dimension of aff_j is $j<n$, $\mu(\text{aff}_j)=0$. Hence, there exists s_{j+1} in $S\setminus\text{aff}_j$ such that $k(s_{j+1})<\infty$, for otherwise k is not in L_p. The points s_i, $0\leqslant i\leqslant j+1$, are then affinely independent.

Now let A denote the convex hull of s_i, $0\leqslant i\leqslant n$. Clearly $\text{int}(A)$ is not empty. If $s\in\text{int}(A)$ then there exist $\lambda_i>0$ such that $\Sigma\lambda_i=1$ and $s=\Sigma_i\lambda_i s_i$. We then have, by convexity, $k(s)\leqslant \Sigma\lambda_i k(s_i)$. Since $k(s_0)=-\infty$ and $k(s_i)<\infty$, $i\geqslant 1$, we have $k(s)=-\infty$ on a set of positive measure. Thus k is not in L_p. The proof is complete.

3. MAIN RESULTS

We establish properties of norm-bounded subsets and convergent sequences of convex functions as well as the existence of a best approximation. If $A, B\subset R^m$, let $\text{dist}(A, B)$ denote $\inf\{|s-t| : s\in A, t\in B\}$. If A is bounded, then there exist u in $\bar{A}$ and v in $\bar{B}$ such that $\text{dist}(A, B)=|u-v|$.

THEOREM 3.1. *Let (k_n) be a sequence of functions in $K\cap L_p$, $1\leqslant p\leqslant\infty$, such that $\|k_n\|_p\leqslant D$ for all n and some $D>0$. Let $T\subset\text{int}(S)$ be a compact set.*

(ii) *If $1\leqslant p<\infty$, then*

$$\sup\{k_n(s) : s\in T, n\geqslant 1\}<\infty.$$

If $p=\infty$, then $k_n(s)\leqslant \|k_n\|_\infty\leqslant D$ for all s in $\text{int}(S)$.

(ii) *If* $1 \leqslant p < \infty$, *then*

$$\inf\{k_n(s) : s \in S, n \geqslant 1\} > -\infty.$$

If $p = \infty$, *then* $k_n(s) \geqslant -\|k_n\|_\infty \geqslant -D$ *for all* s *in* S.

COROLLARY 3.1. *Let* $k \in K \cap L_p$. *Then for* $1 \leqslant p < \infty$, k *is bounded above on* T *and below on* S. *Hence* $|k| < \infty$ *on* $\operatorname{int}(S)$. *If* $p = \infty$, *then* k *is bounded above on* $\operatorname{int}(S)$ *by* $\|k\|_\infty$ *and below on* S *by* $-\|k\|_\infty$. *For all* p, k *is continuous on* $\operatorname{int}(S)$.

Proof. Since, by Lemma 2.1, T is contained in a compact convex body which is a subset of $\operatorname{int}(S)$, it suffices to prove the result when T itself is such a convex body, and we do so. We prove the theorem and the corollary simultaneously.

We show (i) for $1 \leqslant p < \infty$. Let $M_n = \sup\{k_n(s) : s \in T\}$ and $m_n = \inf\{k_n(s) : s \in T\}$. Then $\infty \geqslant M_n \geqslant m_n \geqslant -\infty$. Suppose first that $M_n = m_n$. Then $k_n = M_n$ on T, and if χ is the characteristic function of T, then

$$D \geqslant \|k_n\|_p \geqslant \|k_n \chi\|_p \geqslant |M_n|\, \mu(T)^{1/p}.$$

Thus $|M_n| \leqslant D\mu(T)^{-1/p}$ independent of n.

Now suppose that $M_n > m_n$. Let N be a positive integer. There exist x_n, y_n in T so that $k_n(x_n) \geqslant \min\{M_n, N\} - 1$ and $k_n(x_n) > k_n(y_n)$. By Lemma 2.3, $k_n(y_n) > -\infty$. Consider the line segment defined by the points $z(\lambda) = x_n + \lambda(x_n - y_n)$, $\lambda \geqslant 0$. Using the convexity condition, it may be easily verified that $k_n(z(\lambda))$ is a nondecreasing function of $\lambda \geqslant 0$. For some $0 \leqslant \lambda_1 < \lambda_2$, $z(\lambda_1)$ and $z(\lambda_2)$, lie, respectively, on the boundaries of T and S. Then there exists $\bar{\lambda}$ with $\lambda_1 < \bar{\lambda} < \lambda_2$ such that, if $z_n = z(\bar{\lambda})$, then $d(z_n, T) = d(z_n, R^m \backslash S) = \theta_n$, say, where d is defined in Section 2. Suppose $u \in T$ and $v \in \bar{S} \backslash \operatorname{int}(S)$ such that $|z_n - u| = |z_n - v| = \theta_n$. Let $\operatorname{dist}(T, R^m \backslash S) = c > 0$. Then

$$c \leqslant |u - v| \leqslant |z_n - u| + |z_n - v| = 2\theta_n.$$

Hence, if $0 < r < c/2$, then $B_n = B(z_n, r) \subset \operatorname{int}(S) \backslash T$ for all n. By the monotonicity of $k_n(z(\lambda))$, we have $k_n(z_n) \geqslant k_n(x_n)$. Also $\mu(B_n) = \rho > 0$ independent of n. Let

$$C_n = \{s \in B_n : k_n(s) \geqslant k_n(z_n)\}.$$

We next show that $\mu(C_n) \geqslant \rho/2 > 0$. If $s \in B_n$, then either $s \in C_n$ or $s \in B_n \backslash C_n$. In the latter case, if $z_n = (s + t)/2$ for t in S, then clearly $t \in B_n$. The convexity inequality gives $k_n(z_n) \leqslant (k_n(s) + k_n(t))/2$, i.e.,

$$k_n(t) - k_n(z_n) \geqslant k_n(z_n) - k_n(s) \geqslant 0.$$

Thus, $t \in C_n$, and we have shown that $B_n = C_n \cup \{2z_n - C_n\}$. Hence $\mu(C_n) \geqslant \mu(B_n)/2 \geqslant \rho/2 > 0$. If χ is the characteristic function of C_n, then since $k_n(s) \geqslant k_n(x_n) \geqslant \min\{M_n, N\} - 1$ for s in C_n, we conclude that

$$D \geqslant \|k_n\|_p \geqslant \|k_n \chi\|_p \geqslant \max\{\min\{M_n, N\} - 1, 0\} \, \mu(C_n)^{1/p}$$

for all n and N. Hence, letting $N \to \infty$, we conclude that M_n are bounded above uniformly in n. Thus (i) is established for $1 \leqslant p < \infty$.

Now suppose that $p = \infty$. Let $k \in K \cap L_\infty$. We show that $k(s) \leqslant \|k\|_\infty$ for all s in $\operatorname{int}(S)$. This establishes the assertion. Suppose that there exists t in $\operatorname{int}(S)$ such that $k(t) > \|k\|_\infty$. Then there exists $r > 0$ such that $T = \bar{B}(t, r) \subset \operatorname{int}(S)$. Then by arguing as above for this compact convex body T, we show that there exists $C \subset \operatorname{int}(S)$ with $\mu(C) > 0$ and $k(s) \geqslant k(t) > \|k\|_\infty$ for all s in C. This contradiction to the definition of $\|k\|_\infty$ proves the assertion. The proof of (i) is complete.

Before proceeding to (ii) we establish Corollary 3.1. If $1 \leqslant p < \infty$, we may argue as in (i) or let $k_n = k$ for all n there to conclude that k is bounded above on T. It follows that $k < \infty$ on $\operatorname{int}(S)$. Again, by Lemma 2.3, $k > -\infty$ on $\operatorname{int}(S)$. Thus k is finite on S. Hence k is continuous on $\operatorname{int}(S)$ [6, Theorem 10.1]. Let $u \in \operatorname{int}(S)$. Then there exists a subgradient at u [6, Theorem 23.4]. Hence, k is bounded below on S. If $p = \infty$, then, as in (i), $k(s) \leqslant \|k\|_\infty$ for all s in $\operatorname{int}(S)$. Also, we may show as above that k is continuous on $\operatorname{int}(S)$. Since $k \geqslant -\|k\|_\infty$ a.e. on S, continuity of k shows that $k(s) \geqslant -\|k\|_\infty$ for all s in $\operatorname{int}(S)$. Now let $t \in S \setminus \operatorname{int}(S)$ and $s \in \operatorname{int}(S)$. Then $\lambda t + (1-\lambda) s$ is in $\operatorname{int}(S)$ for all $0 \leqslant \lambda < 1$ [6, Theorem 6.1]. Hence, by Theorem 7.5 of [6], we have $k(t) \geqslant \lim k(\lambda t + (1-\lambda) s)$ as $\lambda \uparrow 1$. (In the notation of that theorem, we have $f(y) \geqslant (\operatorname{cl} f)(y)$.) Hence $k(t) \geqslant -\|k\|_\infty$. The corollary is now established.

Now we establish (ii) for $1 \leqslant p < \infty$. As shown above, each k_n is finite and continuous on $\operatorname{int}(S)$. We first show that (k_n) is bounded below on $\operatorname{int}(S)$ uniformly for all n. Assume to the contrary that there exist t_n in $\operatorname{int}(S)$ with $k_n(t_n) < 0$ for all n and $k_n(t_n) \to -\infty$. We reach a contradiction. Let

$$F_n = \{s \in \operatorname{int}(S) : k_n(s) < 0\}$$

and $k_n(t_n) = c_n$. Then $t_n \in F_n$ and F_n is open by continuity of k_n on $\operatorname{int}(S)$. We show that $\mu(F_n) \to 0$. Define $V_n \subset R^{m+1}$ by

$$V_n = \operatorname{co}\{\{(s, 0) : s \in F_n\} \cup \{(t_n, c_n)\}\},$$

where $\operatorname{co}(A)$ denotes the convex hull of $A \subset R^{m+1}$. It is easy to see that V_n is a convex cone with base $\{(s, 0) : s \in F_n\}$ and apex (t_n, c_n). Clearly, the epigraph $E(k_n)$ of k_n, because of its convexity, contains V_n. Define a function f_n on F_n by

$$f_n(s) = \inf\{u : (s, u) \in V_n\}, \; s \in F_n.$$

Then by convexity of V_n, f_n is convex on F_n, $0 \geqslant f_n \geqslant k_n$ on F_n, and $f_n(t_n) = c_n$. Now let

$$H_n = \{s \in F_n : f_n(s) \leqslant c_n/2\}.$$

Then $H_n \subset F_n$ and $\mu(H_n) = \mu(F_n)/2^m$. Define a function h_n on S by $h_n(s) = c_n/2$, if $s \in H_n$, and 0, otherwise. Then $|k_n| \geqslant |h_n|$ on S, and we have

$$D \geqslant \|k_n\|_p \geqslant \|h_n\|_p = (-c_n/2)(\mu(F_n)/2^m)^{1/p}.$$

Since $c_n \to -\infty$, we have $\mu(F_n) \to 0$.

We now apply Lemma 2.2. If d and δ_n are as defined there, we conclude that $\liminf \delta_n > 0$. For convenience of notation, assume $\lim \delta_n = 4\theta > 0$. Choose N so that $\delta_n \geqslant 2\theta$ for $n \geqslant N$. In what follows consider $n \geqslant N$. There exists $u_n \in \bar{S} \backslash F_n$ such that $\delta_n = d(u_n)$. Since $\delta_n > 0$, $u_n \in \operatorname{int}(S)$. Consider the line segment L joining t_n and u_n. It intersects the boundary of F_n at x_n. When L is extended beyond u_n, it intersects the boundary of S at z_n. Clearly $d(z_n) = 0$. Since t_n and u_n are in $\operatorname{int}(S)$, which is convex, we conclude that x_n is in $\operatorname{int}(S)$. By continuity of k_n on $\operatorname{int}(S)$, we have $k_n(x_n) = 0$. Let $y_n = (u_n + z_n)/2$. Then, by concavity of d, we have

$$d(y_n) \geqslant (d(u_n) + d(z_n))/2 \geqslant \delta_n/2 \geqslant \theta.$$

Hence, by Proposition 2.1, y_n lies in the compact set $T = \{s \in S : d(s) \geqslant \theta\}$ and $T \subset \operatorname{int}(S)$. Also,

$$|y_n - x_n| \geqslant |y_n - u_n| = |z_n - u_n|/2 \geqslant \delta_n/2 \geqslant \theta.$$

Now $x_n = \lambda_n t_n + (1 - \lambda_n)\, y_n$ for some $0 \leqslant \lambda_n \leqslant 1$. By the above observation we must have $\lambda_n \geqslant \rho > 0$ for some ρ. Now,

$$0 = k_n(x_n) \leqslant \lambda_n k_n(t_n) + (1 - \lambda_n)\, k_n(y_n),$$

which gives $k_n(y_n) \geqslant -\lambda_n/(1 - \lambda_n)\, k_n(t_n)$. Since $k_n(t_n) \to -\infty$, we conclude that $k_n(y_n)$ are not bounded above. Again, since $\{y_n\} \subset T$, this is a contradiction to (i). Thus, (k_n) is bounded below on $\operatorname{int}(S)$ uniformly in n.

Now, if $t \in S \backslash \operatorname{int}(S)$, then, as in the above proof of Corollary 3.1, we let $s \in \operatorname{int}(S)$ and observe $k_n(t) \geqslant \lim k_n(\lambda t + (1 - \lambda)\, s)$ as $\lambda \uparrow 1$, where $\lambda t + (1 - \lambda)\, s$ is in $\operatorname{int}(S)$ for all $0 \leqslant \lambda < 1$. This shows that (k_n) is bounded below on S uniformly in n. The proof for $1 \leqslant p < \infty$ is complete. Now the proof for $p=\infty$ is contained in the proof of Corollary 3.1. The proof is complete.

A function f in H is said to be lower semi-continuous at s in S if $f(s) \leqslant \lim f(s_i)$ for every sequence (s_i) in S such that s_i converges to s and the limit of $(f(s_i))$ exists in $[-\infty, \infty]$.

THEOREM 3.2. *Let (k_n) be a sequence of functions in $K \cap L_p$, $1 \leqslant p \leqslant \infty$, such that $\|k_n\|_p \leqslant D$ for all n and some $D>0$. Then there exists a subsequence (g_i) of (k_n) and a g in $K \cap L_p$ such that $g_i \rightarrow g$ pointwise on* $\operatorname{int}(S)$, *and hence a.e. on S, since $\mu(S \setminus \operatorname{int}(S))=0$. Such a g has the following properties: g is lower semi-continuous on S, $|g|<\infty$ on* $\operatorname{int}(S)$, *and $\|g\|_p \leqslant D$. Furthermore, the convergence of g_i to g is uniform on every compact $T \subset \operatorname{int}(S)$.*

Proof. We prove the result for $1 \leqslant p<\infty$. The proof for $p=\infty$ is simpler. By Theorem 3.1, the real number sequence $(k_n(s))$ is bounded for each s in $\operatorname{int}(S)$. Since $\operatorname{int}(S)$ is relatively open, by Theorem 10.9 of [6], there exists a finite convex function g on $\operatorname{int}(S)$ and a subsequence (g_i) of (k_n) such that $g_i \rightarrow g$ pointwise on $\operatorname{int}(S)$ and uniformly on a compact T. To extend g to S let $t \in S \setminus \operatorname{int}(S)$ and $s \in \operatorname{int}(S)$. Then $\lambda t+(1-\lambda) s \in \operatorname{int}(S)$ for all $0 \leqslant \lambda<1$, and we set

$$g(t)=\operatorname{limit} g(\lambda t+(1-\lambda) s), \qquad t \uparrow 1.$$

Then, by Theorem 7.5 of [6], g is lower semi-continuous on S. (Note that in that theorem, $\operatorname{cl} f$ is lower semi-continuous.) Such an extension is independent of the choice of s.

It now suffices to show that $g \in L_p$ with $\|g\|_p \leqslant D$. Indeed, let (T_n) be a sequence of compact convex sets with $T_n \subset T_{n+1}$ and $\bigcup T_n=\operatorname{int}(S)$. (Lemma 2.1 gives a procedure for constructing such a sequence.) Let χ_n be the characteristic function of T_n. By Theorem 3.1, there exists a finite positive number M_n such that $|g_i \chi_n| \leqslant M_n$. Since constant functions are in L_p, using the bounded convergence theorem [3], we let $i \rightarrow \infty$ in the obvious inequality $\|g_i \chi_n\|_p \leqslant \|g_i\|_p \leqslant D$ and conclude that $\|g \chi_n\|_p \leqslant D$. Now $|g \chi_n|^p \uparrow |g|^p$ on $\operatorname{int}(S)$ as $n \rightarrow \infty$. Hence, by the monotone convergence theorem [3], we have $\|g\|_p \leqslant D$. The proof is complete.

THEOREM 3.3. (i) *Suppose that $P \subset K$ satisfies $P \cap L_p=\bar{P} \cap L_p$, $1 \leqslant p \leqslant \infty$. Then $P \cap L_p$ is closed in L_p and a best approximation to f in L_p from $P \cap L_p$ exists. In particular, $K \cap L_p$ is a closed convex cone and a best approximation from $K \cap L_p$ exists.*

(ii) *Let (k_n) be a sequence in $K \cap L_p$, $1 \leqslant p<\infty$, such that*

$\|k_n - k\|_p \to 0$ *for some* k *in* L_p *which is continuous on* $\operatorname{int}(S)$. *Then* $k_n \to k$ *pointwise on* $\operatorname{int}(S)$ *and uniformly on every compact* $T \subset \operatorname{int}(S)$.

Proof. (i) By taking convergent sequences it is easy to show that $K = \bar{K}$, and hence $K \cap L_p = \bar{K} \cap L_p$. Now Theorem 3.2 shows that $P \cap L_p$ and $K \cap L_p$ satisfy the conditions of Proposition 2.2. Hence the assertions follow.

(ii) By Lemma 2.1, we may assume that T is a compact convex body. By Theorem 3.1, (k_n) is bounded on T uniformly in n. Hence, by Theorem 10.6 of [6], (k_n) is equi-Lipschitzian relative to T. Now Proposition 2.3 shows that $k_n \to k$ uniformly on T. Since $T \subset \operatorname{int}(S)$ is arbitrary, this implies that $k_n \to k$ on $\operatorname{int}(S)$. The proof is complete.

The existence and uniqueness of a best approximation from $K \cap L_p$, $1 < p < \infty$, also follows from the uniform convexity of L_p, $1 < p < \infty$, and the closedness and the convexity of $K \cap L_p$.

We now present an alternative approach to the analysis of our problem. By Lemma 2.3, if $k \in L_p$ is convex, then $k > -\infty$ on S. Using this fact, we may give another definition of a convex function: k in H is convex if $k > -\infty$ on S and the convex inequality (1.1) holds for all s, t in S. Clearly, the terms $\infty - \infty$ cannot appear in this definition. Let K_1 be the set of all so defined convex functions. Note that $K_1 \subset \bar{K}_1 = K$. (To show $\bar{K}_1 = K$, let $k \in K$. Then $k_n = \max\{k, -n\}$ is in K_1 for all n and $k_n \to k$.) The following lemma may be established by methods similar to that of Lemma 2.3.

LEMMA 3.1.

$$K_1 \cap L_p = \bar{K}_1 \cap L_p, \qquad 1 \leqslant p \leqslant \infty.$$

It follows that $K_1 \cap L_p = K \cap L_p$. All the results of Section 3 remain valid if we replace K there by K_1.

REFERENCES

1. E. DEAK, Über konvexe und interne Funktionen, sowie eine gemeinsame Verallgemeinerung von beiden, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **5** (1962), 109–154.
2. F. DEUTSCH, Existence of best approximations, *J. Approx. Theory* **28** (1980), 132–154.
3. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators, Part I," Interscience, New York, 1958.
4. D. LANDERS AND L. ROGGE, Isotonic Approximation in L_s, *J. Approx. Theory* **31** (1981), 199–223.
5. J. T. LEWIS AND O. SHISHA, L_p convergence of monotone functions and their uniform convergence, *J. Approx. Theory* **14** (1975), 281–284.

6. R. T. ROCKAFELLAR, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
7. V. A. UBHAYA, Uniform approximation by quasi-convex and convex functions, *J. Approx. Theory* **55** (1988), 326–336.
8. V. A. UBHAYA, L_p approximation from nonconvex subsets of special classes of functions, *J. Approx. Theory* **57** (1989), 223–238.
9. V. A. UBHAYA, L_p approximation by quasi-convex and convex functions, *J. Math. Anal. Appl.* **139** (1989), 574–585.
10. V. A. UBHAYA, Lipschitzian selections in best approximation by continuous functions, *J. Approx. Theory* **61** (1990), 40–52.
11. E. ZEIDLER, "Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization, Springer-Verlag, New York, 1985.
12. F. A. VALENTINE, "Convex Sets," Krieger, Huntington, NY, 1976.